A SIMPLE PROOF OF THE AFFABILITY THEOREM FOR PLANAR TILINGS

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Abstract. We give a simple proof of the Affability Theorem of T. Giordano, H. Matui, I. Putnam and C. Skau for aperiodic and repetitive planar tilings.

1. Introduction

A *planar tiling* is a partition of $\mathbb{R}^2$ into *tiles*, which are polygons touching face-to-face obtained by translation from a finite set of *prototiles*. Such a tiling always satisfies the *Finite Pattern Condition* given in [3]. It is said to be *aperiodic* if it has no translation symmetries, and *repetitive* if for any patch M, there exists a constant $R > 0$ such that any ball of radius R contains a translated copy of M.

Let $\mathbb{T}(\mathcal{P})$ be the set of tilings $\mathcal{T}$ obtained from a finite set of prototiles $\mathcal{P}$. It is possible to endow $\mathbb{T}(\mathcal{P})$ with the *Gromov-Hausdorff topology* [3, 4] generated by the basic neighborhoods

$$U^r_{\varepsilon,\varepsilon'} = \{\, \mathcal{T}' \in \mathbb{T}(\mathcal{P}) \mid \exists\, v, v' \in \mathbb{R}^2 : \|v\| < \varepsilon, \|v'\| < \varepsilon', R(\mathcal{T}+v, \mathcal{T}'+v') > r \,\}$$

where $R(\mathcal{T},\mathcal{T}')$ is the supremum of radii $R > 0$ such that $\mathcal{T}$ and $\mathcal{T}'$ coincide on the ball $B(0,R)$. Then $\mathbb{T}(\mathcal{P})$ becomes a compact metrizable space, which is naturally laminated by the orbits $L_{\mathcal{T}}$ of the action of the $\mathbb{R}^2$ by translations. For each $\mathcal{T} \in \mathbb{T}(\mathcal{P})$, we denote by $D_{\mathcal{T}}$ the Delone set determined by the choice of base points in the prototiles. Now $T = \{\, \mathcal{T} \in \mathbb{T}(\mathcal{P}) \mid 0 \in D_{\mathcal{T}} \,\}$ is a totally disconnected closed subspace which meets all the leaves, so T is a *total transversal* for $\mathbb{T}(\mathcal{P})$.

If $\mathcal{T} \in \mathbb{T}(\mathcal{P})$ is a repetitive tiling, then $\mathbb{X} = \overline{L}_{\mathcal{T}}$ is a minimal closed subset of $\mathbb{T}(\mathcal{P})$, called the *continuous hull of* $\mathcal{T}$. If $\mathcal{T}$ is also aperiodic, any tiling in $\mathbb{X}$ has the same property and therefore $X = T \cap \mathbb{X}$ is homeomorphic to the Cantor set. In this case, the lamination $\mathcal{F} = \{L_{\mathcal{T}}\}_{\mathcal{T}\in\mathbb{X}}$ of $\mathbb{X}$ induces an étale equivalence relation (EER) $\mathcal{R} = \{\, (\mathcal{T}, \mathcal{T} - v) \in X \times X \mid v \in D_{\mathcal{T}} \,\}$ on X which completely represents its transverse dynamics.

One of the first and most remarkable examples in the theory of aperiodic tilings was constructed by R. M. Robinson in [14] from a set of 32 aperiodic prototiles, 6 up to isometries of the plane. For any repetitive Robinson tiling $\mathcal{T}$, there is a Borel isomorphism between the space of sequences of 0s and 1s equipped with the cofinal or tail equivalence relation and a full measure and residual subset of the

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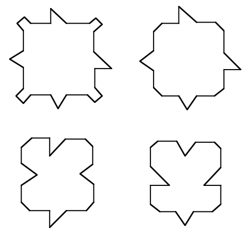


Figure 1. Robinson's aperiodic tiles

total transversal X of its continuous hull $\mathbb{X}$. Let us recall that all cofinality classes are orbits of a 2-adic adding machine, except for one orbit that is the union of two cofinality classes. We say that the transverse dynamics of the space of repetitive Robinson tilings is represented (in a measurable way) by a 2-adic adding machine. Better yet, the EER $\mathcal{R}$ induced on the total transversal X is topologically orbit equivalent to a minimal dynamical system on the Cantor set. In fact, T. Giordano, I. Putnam and C. Skau showed in [8] that all equivalence relations arising from this type of dynamical systems are topologically orbit equivalent to AF equivalence relations, i.e. given as an inductive limit of finite equivalence subrelations. According to [9], an equivalence relation is said to be *affable* if it is orbit equivalent to an AF equivalence relation. Extending an earlier result of H. Matui [12] for a class of substitution tilings, these authors have proved that all aperiodic and repetitive planar tilings (or equivalently minimal $\mathbb{Z}^2$-actions on Cantor sets) have the same affability property than Robinson repetitive tilings:

Theorem 1.1 (**Affability Theorem**, [5]). *The continuous hull of any aperiodic and repetitive planar tiling is affable.*

More generally, an Affability Theorem for Cantor minimal $\mathbb{Z}^m$-systems has been proved in [7] by the same authors. In both cases, in order to demonstrate the affability of the corresponding equivalence relations, they combine combinatorial methods and convexity arguments with an important result regarding the extension of minimal AF equivalence relations, called the Absorption Theorem [6]. In [1], we announced another proof (where we used a former version of the Absorption Theorem), but there are two gaps in the proof of Lemma 3.1 and Theorem 5.1. In this paper, we close these gaps by defining a special inflation process, which is similar to that used to construct Robinson tilings. We hope that this approach, where no convexity argument is needed, helps to achieve the declared goal by Giordano, Putnam and Skau (see [9]) of proving that any countable amenable group acting minimally and freely on a totally disconnected compact space defines an affable equivalence relation. In fact, our method remains valid for aperiodic and repetitive tilings of $\mathbb{R}^m$, although details will be published elsewhere.

2. Affable equivalence relations

Let $\mathcal{R}$ be an EER on a locally compact space X. Following [9], we say that $\mathcal{R}$ is a *compact étale equivalence relation* (CEER) if $\mathcal{R} - \Delta_X$ is a compact subset of $X \times X$ (where Δ_X is the diagonal of $X \times X$). This means that $\mathcal{R}$ is *proper* in the sense of [13] and trivial out of a compact set.

Definition 2.1 ([9]). An equivalence relation $\mathcal{R}$ on a totally disconnected space X is *affable* if there exists an increasing sequence of CEERs $\mathcal{R}_n$ such that $\mathcal{R} = \bigcup_{n\in\mathbb{N}} \mathcal{R}_n$. The inductive limit topology turns $\mathcal{R}$ into an EER and we say that $\mathcal{R} = \varinjlim \mathcal{R}_n$ is *approximately finite* (or AF).

An example of AF equivalence relation is the cofinal equivalence relation on the infinite path space of a particular type of oriented graphs (V, E), called *Bratteli diagrams*. According to [8] and [9], their vertices are stacked on levels and their edges join two consecutive levels. More precisely, we denote by V_n the set of vertices of the level n and by E_n the set of edges e with origin $s(e)$ in V_{n-1} and endpoint $r(e)$ in V_n so that $V = \bigcup V_n$ and $E = \bigcup E_n$. In fact, they are actually the only examples of AF equivalence relations:

Theorem 2.2 ([9], [13]). *Let $\mathcal{R}$ be an AF equivalence relation on a totally disconnected space X. There exists a Bratteli diagram (V, E) such that $\mathcal{R}$ is isomorphic to the tail equivalence relation on the infinite path space*

$$X_{(V,E)} = \{ (e_1, e_2, \dots) \mid e_i \in E_i, r(e_i) = s(e_{i+1}), \forall i \geq 1 \}$$

given by

$$\mathcal{R}_{cof} = \{ ((e_1, e_2, \dots), (e'_1, e'_2, \dots)) \in X_{(V,E)} \times X_{(V,E)} \mid \exists\, m \geq 1 : e_n = e'_n, \forall\, n \geq m \}.$$

If X is compact, then (V, E) can be chosen standard*, i.e. $V_0 = \{v_0\}$ and $r^{-1}(v) \neq \emptyset$ for all $v \in V - \{v_0\}$. Furthermore, $\mathcal{R}$ is minimal if and only if (V, E) is* simple*, i.e. for each $v \in V$, there exists $m \geq 1$ such that all vertices in V_m are reachable from v.* □

As for the continuous hull of a Robinson repetitive tiling, the cofinal equivalence relation on the infinite path space of a simple *ordered* Bratteli diagram (i.e. having a linear order on each set of edges with the same endpoint) is essentially isomorphic to a Cantor minimal $\mathbb{Z}$-system. Indeed, by using lexicographic order on cofinal infinite paths and sending the unique maximal path to the unique minimal path, we have a minimal homeomorphism $\lambda_{(V,E)} : X_{(V,E)} \to X_{(V,E)}$, called a *Vershik map*, see [8] and [9]. This map preserves cofinality, except for the maximal and minimal paths. We refer to the corresponding dynamical system as the *Bratteli-Vershik $\mathbb{Z}$-system* associated to (V, E). Now, as we can see in [9], it is not difficult to prove that any minimal AF equivalence relation on the Cantor set X is orbit equivalent to its Bratteli-Vershik $\mathbb{Z}$-system.

3. The global schema

In this section, we describe the general steps of the proof (as explained in [1]):

- The first step consist of applying the *inflation* or *zooming process* developed in [3] to obtain an increasing sequence of CEERs $\mathcal{R}_n$, and thus an open AF equivalence subrelation $\mathcal{R}_\infty = \varinjlim \mathcal{R}_n$ of $\mathcal{R}$.
- In the second step, we define a *discrete boundary $\partial\mathcal{R}_\infty$ of $\mathcal{R}_\infty$* and we study its properties. It is a nonempty meager closed subset of X whose saturation contains all the points with $\mathcal{R}_\infty$-equivalence class different from its $\mathcal{R}$-equivalence class.
- In order to apply the *Absorption Theorem* of [6] in the third step, we must prove that the discrete boundary $\partial\mathcal{R}_\infty$ is $\mathcal{R}_\infty$*-thin* in the sense of [9]. This means that $\mu(\partial\mathcal{R}_\infty) = 0$ for every $\mathcal{R}_\infty$-invariant probability measure μ. But it will also be

essential to verify that all $\mathcal{R}$-equivalence classes split into a (uniformly bounded) finite number of $\mathcal{R}_\infty$-equivalence classes.

In Section 4, we present a special inflation process, which we call *Robinson inflation*, allowing us to construct a sequence of transverse CEERs with all of the necessary properties to apply absorption techniques from [6].

3.1. **Inflation.** We start by recalling the general *inflation process* developed in [3]. By definition of its topology, the continuous hull $\mathbb{X}$ admits a *box decomposition* $\mathcal{B} = \{\mathbb{B}_i\}_{i=1}^k$ consisting of closed flow boxes $\varphi_i : \mathbb{B}_i \to \mathbb{P}_i \times X_i$ such that $\mathbb{X} = \bigcup_{i=1}^k \mathbb{B}_i$ and $\mathring{\mathbb{B}}_i \cap \mathring{\mathbb{B}}_j = \emptyset$ if $i \neq j$. In this context, we can also assume that the plaque $\mathbb{P}_i$ is a $\mathcal{P}$-tile and the change of coordinates is given by

$$\varphi_i \circ \varphi_j^{-1}(x, y) = \big(\varphi_{ij}(x), \gamma_{ij}(y)\big)$$

where the map φ_{ij} is a translation from an edge of $\mathbb{P}_j$ to an edge of $\mathbb{P}_i$. Nevertheless, the laminated space $\mathbb{X}$ admits more general box decompositions as defined in [2]. In general, any box decomposition is said to be *well-adapted* (to the $\mathcal{P}$-tiled structure) if each plaque $\mathbb{P}_i$ is a $\mathcal{P}$-patch and the associated total transversal $\bigsqcup_{i=1}^k X_i$ is a clopen subset of X. For any flow box $\mathbb{B}_i$ in $\mathcal{B}$, the set $\partial_v \mathbb{B}_i = \varphi_i^{-1}(\partial \mathbb{P}_i \times X_i)$ is called the *vertical boundary* of $\mathbb{B}_i$.

Theorem 3.1 ([3])**.** *Let $\mathbb{X}$ be the continuous hull of an aperiodic and repetitive Euclidean tiling satisfying the finite pattern condition. Then, for any well-adapted flow box decomposition $\mathcal{B}$ of $\mathbb{X}$, there exists another well-adapted flow box decomposition $\mathcal{B}'$* inflated from $\mathcal{B}$ *having the following properties:*

i) for each tiling $\mathcal{T}$ in a box $\mathbb{B} \in \mathcal{B}$ and in a box $\mathbb{B}' \in \mathcal{B}'$, the transversal of $\mathbb{B}'$ through $\mathcal{T}$ is contained in the corresponding transversal of $\mathbb{B}$;

ii) the vertical boundary of the boxes of $\mathcal{B}'$ is contained in the vertical boundary of boxes of $\mathcal{B}$;

iii) for each box $\mathbb{B}' \in \mathcal{B}'$, there exists a box $\mathbb{B} \in \mathcal{B}$ such that $\mathbb{B} \cap \mathbb{B}' \neq \emptyset$ and $\mathbb{B} \cap \partial_v \mathbb{B}' = \emptyset$. □

By applying this theorem inductively, we have a sequence of well-adapted box decompositions $\mathcal{B}^{(n)}$ such that

1) $\mathcal{B}^{(0)} = \mathcal{B}$,
2) $\mathcal{B}^{(n+1)}$ is inflated from $\mathcal{B}^{(n)}$ and
3) $\mathcal{B}^{(n+1)}$ defines a finite set $\mathcal{P}^{(n+1)}$ of $\mathcal{P}^{(n)}$-patches (which contain at least a $\mathcal{P}^{(n)}$-tile in their interiors) and a tiling in $\mathbb{T}(\mathcal{P}^{(n+1)})$ of each leaf of $\mathbb{X}$.

As $\mathcal{P}^{(n+1)}$-tiles are $\mathcal{P}^{(n)}$-patches and also plaques of $\mathcal{B}^{(n+1)}$, we will always use is the same letter $\mathbb{P}$ to denote them.

Let $X^{(n)}$ be the total transversal associated to $\mathcal{B}^{(n)}$. Given any increasing sequence of integers N_n, we can construct a sequence of inflated box decompositions $\mathcal{B}^{(n)}$ that such each Delone set $D_{\mathcal{T}}^{(n)} = L_{\mathcal{T}} \cap X^{(n)}$ is N_n-separated. Such a sequence defines an increasing sequence of CEERs $\mathcal{R}_n$ on X. Indeed, for each $n \in \mathbb{N}$, the equivalence class $\mathcal{R}_n[\mathcal{T}]$ coincides with the *discrete plaque* $P_n = \mathbb{P}_n \cap X$ determined by the plaque $\mathbb{P}_n$ of $\mathcal{B}^{(n)}$ passing through $\mathcal{T}$. We can also see each discrete plaque P_n as a plaque of a *discrete flow box* $B = \mathbb{B} \cap X$ and each discrete flow box B as an element of a discrete box decomposition defined by $\mathcal{B}$.

Proposition 3.2. *The inductive limit $\mathcal{R}_\infty = \varinjlim \mathcal{R}_n$ is a minimal open AF equivalence subrelation of $\mathcal{R}$.*

Proof. It is clear that $\mathcal{R}_\infty$ is an open AF equivalence subrelation of $\mathcal{R}$. On the other hand, in order to show that $\mathcal{R}_\infty$ is minimal, we must prove that all $\mathcal{R}_\infty$-equivalence classes meet any open subset A of X. But each $\mathcal{R}_\infty$-equivalence class contains an increasing sequence of discrete plaques $P_n = \mathbb{P}_n \cap X$ where $\mathbb{P}_n$ are the plaques of the box decompositions $\mathcal{B}^{(n)}$. Since $\mathcal{R}$ is minimal, the intersection $A \cap L_{\mathcal{T}}$ remains a Delone set quasi-isometric to $L_{\mathcal{T}}$, and therefore $A \cap \mathbb{P}_n = A \cap P_n \neq \emptyset$ for some $n \in \mathbb{N}$. Then $A \cap \mathcal{R}_\infty[\mathcal{T}]$ is also nonempty, and thus $\mathcal{R}_\infty[\mathcal{T}]$ is dense. □

Remark 3.3. The inflation process developed in [3] for tilings and tilable laminations has been extended in [2] and [11] for transversely Cantor laminations.

3.2. **Boundary.** Let us start by defining the (discrete) boundary of a EER $\mathcal{R}_n$:

Definition 3.4. i) For any tiling $\mathcal{T} \in X$, let $\partial\mathcal{R}_n[\mathcal{T}]$ be the set of tilings $\mathcal{T}' = \mathcal{T} - v$ such that v is the base point of any $\mathcal{P}$-tile of the patch $\mathbb{P}_n$ meeting $\partial\mathbb{P}_n$.

ii) We define the *boundary of* $\mathcal{R}_n$ as the clopen set

$$\partial\mathcal{R}_n = \bigcup_{\mathcal{T} \in X^{(n)}} \partial\mathcal{R}_n[\mathcal{T}] = \bigcup_{\mathbb{B}_n \in \mathcal{B}^{(n)}} \partial_v B_n$$

where $\partial_v B_n$ is the vertical boundary of the discrete flow box $B_n = \mathbb{B}_n \cap X$.

iii) Finally, $\partial\mathcal{R}_\infty = \bigcap_{n\in\mathbb{N}} \partial\mathcal{R}_n$ is a meager closed subset of X, which we shall call the *boundary of* $\mathcal{R}_\infty$.

Notice that the open AF equivalence subrelation $\mathcal{R}_\infty$ is not maximal since, even if $\mathcal{R}$ is AF, its boundary $\partial\mathcal{R}_\infty$ is always nonempty. For each tiling $\mathcal{T} \in \partial\mathcal{R}_\infty$, $\mathcal{R}[\mathcal{T}]$ separates into several $\mathcal{R}_\infty$-equivalence classes. By replacing the elements of theses $\mathcal{R}_\infty$-equivalence classes with the corresponding $\mathcal{P}$-tiles, we obtain a decomposition of the leaf $L = L_{\mathcal{T}}$ passing through $\mathcal{T}$ into regions having a common boundary $\Gamma = \Gamma_{\mathcal{T}}$. This boundary is a union of edges of $\mathcal{T}$. The union of all these common boundaries is a closed subset

$$\partial_c\mathcal{R}_\infty = \bigcap_{n\in\mathbb{N}} \bigcup_{\mathbb{B}_n \in \mathcal{B}^{(n)}} \partial_v \mathbb{B}_n$$

of $\mathbb{X}$, which we call the *continuous boundary* of $\mathcal{R}_\infty$. For each $n \geq 1$, the closed subset $\bigcup_{\mathbb{B}_n \in \mathcal{B}^{(n)}} \partial_v \mathbb{B}_n$ of $\mathbb{X}$ intersects the leaf L into an infinite graph $\Gamma^{(n)}$ where each edge separates two different tiles of $\mathcal{T}$ and is never terminal. According to property (*iii*) in Theorem 3.1, $\Gamma = \bigcap_{n\in\mathbb{N}} \Gamma^{(n)}$ is an infinite acyclic graph without terminal edges. As the box decomposition $\mathcal{B}^{(0)} = \mathcal{B}$ is finite, $\Gamma^{(0)}$ has *bounded geometry*, i.e. each vertex has uniformly bounded degree. The same happens with Γ and $\Gamma^{(n)}$ for all $n \in \mathbb{N}$. Moreover, by construction, the number of connected components of $L - \Gamma$ is equal to the number of $\mathcal{R}_\infty$-equivalence classes in $\mathcal{R}[\mathcal{T}]$, but this number may be infinite.

By replacing X with a different total transversal $\check{X}$ passing through the boundary of each tile, we can assume that the corresponding discrete boundary $\check{\partial}\mathcal{R}_\infty$ is equal to $\partial_c\mathcal{R}_\infty \cap \check{X}$. However, each point in the new boundary $\check{\partial}\mathcal{R}_\infty$ determines several points (of the interior of different adjacent tiles) in the original boundary $\partial\mathcal{R}_\infty$. We resume the above discussion in the following result:

Proposition 3.5. *The continuous boundary $\partial_c\mathcal{R}_\infty$ is a closed subset of $\mathbb{X}$ which admits a natural partition into acyclic graphs without terminal edges (induced by the lamination of $\mathbb{X}$) with total tranversal $\check{\partial}\mathcal{R}_\infty$.*

In order to apply the Absortion Theorem of [6] in the next step, we also need to prove the following result:

Proposition 3.6. *The boundary $\partial\mathcal{R}_\infty$ is $\mathcal{R}_\infty$-thin.*

In [1], we showed how to derive this result from the type of growth of the leaves of $\mathcal{F}$. To do so, we simply adapted the method used by C. Series in [16] to prove that any measurable foliation with polynomial growth is hyperfinite. But as in the original proof of the Affability Theorem in [5], Proposition 3.6 will be recovered here using the isoperimetric properties of the pieces involved in the Robinson inflation.

3.3. **Absorption.** We describe now the last step of the proof of the Affability Theorem. We start by recalling the Absorption Theorem 4.6 of [6], which is a key ingredient in this proof:

Theorem 3.7 ([6])**.** *Let $\mathcal{R}$ be a minimal AF equivalence relation on the Cantor set X, and let Y be a closed $\mathcal{R}$-thin subset of X such that $\mathcal{R}|_Y$ remains étale. Let $\mathcal{K}$ be a CEER on Y which is transverse to $\mathcal{R}|_Y$ (i.e. $\mathcal{R}|_Y \cap \mathcal{K} = \Delta_Y$ and there is an isomorphism of topological groupoids $\varphi : \mathcal{R}|_Y * \mathcal{K} \to \mathcal{K} * \mathcal{R}|_Y$). Then there is a homeomorphism $h : X \to X$ such that*

i) h implements an orbit equivalence between the equivalence relation $\mathcal{R}\vee\mathcal{K}$ generated by $\mathcal{R}$ and $\mathcal{K}$, and the AF equivalence relation $\mathcal{R}$;

ii) $\mathcal{R}|_{h(Y)}$ is étale and $h(Y)$ is $\mathcal{R}$-thin;

iii) $h|_Y$ implements an isomorphism between $\mathcal{R}|_Y \vee \mathcal{K}$ and $\mathcal{R}|_{h(Y)}$.

In particular, $\mathcal{R}\vee\mathcal{K}$ is affable. □

According to Propositions 3.2, 3.5 and 3.6, the equivalence relation $\mathcal{R}_\infty$ fulfills the initial hypotheses of the Absorption Theorem. On the other hand, the graph of $\mathcal{R}|_{\partial\mathcal{R}_\infty}$ is the union of a countable family of clopen bisections G_i where each G_i is the graph of a partial transformation $\varphi_i : A_i \to B_i$ between disjoint clopen subsets A_i and B_i of $\partial\mathcal{R}_\infty$. Thus, each closed bisection $\bar{G}_i = G_i - \mathcal{R}_\infty$ is the graph of a partial transformation $\bar{\varphi}_i : \bar{A}_i \to \bar{B}_i$ between disjoint closed subsets $\bar{A}_i$ and $\bar{B}_i$ of $\partial\mathcal{R}_\infty$. If we assume that all $\mathcal{R}$-equivalence classes split into at most two $\mathcal{R}_\infty$-equivalence classes, then $\bar{\varphi}_i$ implements an isomorphism between $\mathcal{R}_\infty|_{\bar{A}_i}$ and $\mathcal{R}_\infty|_{\bar{B}_i}$ and its graph $\bar{G}_i$ generates a CEER transverse to $\mathcal{R}_\infty|_{\bar{A}_i\cup\bar{B}_i}$. Then we have a global decomposition $\partial\mathcal{R}_\infty = A \cup B$ and a global transformation $\varphi : A \to B$ whose graph generates a CEER $\mathcal{K}$ transverse to $\mathcal{R}_\infty|_{\partial\mathcal{R}_\infty}$. In fact, in this case, we can use a former version of the Absorption Theorem (Theorem 4.18 of [9]) to see that $\mathcal{R} = \mathcal{R}_\infty \vee \mathcal{K}$ is orbit equivalent to $\mathcal{R}_\infty$. However, in the general case, we cannot divide the boundary $\partial\mathcal{R}_\infty$ into closed pieces with the same finite number of $\mathcal{R}_\infty$-equivalence classes. It is necessary to use some special inflation process like that described in [5] using convexity arguments, or the one we describe later on inspired by Robinson tilings.

4. Robinson inflation

Here we define an inflation process for planar tilings which is modelled by the natural inflation of Robinson tilings (see [10] and [14]). We start by replacing the tiling space with a tiling space whose tiles are marked squares according to a theorem by L. Sadun and R. F. Williams [15]. Later, we construct a family of inflated flow boxes whose plaques are squares of side N_1, that is maximal in the sense that there is no space for any other square of side N_1. Finally, we replace this partial box decomposition with a complete box decomposition in such a way that the isoperimetric ratios of the plaques are still good. In this way, we recurrently obtain an inflation process such that all $\mathcal{R}$-equivalence classes split at most into 4 different $\mathcal{R}_\infty$-equivalence classes. Then the boundary $\partial\mathcal{R}_\infty$ can be filtered by a countable number of $\mathcal{R}_\infty$-thin closed subsets equipped with CEERs transverse to $\mathcal{R}_\infty|_{\partial\mathcal{R}_\infty}$.

4.1. **Sadun-Williams reduction to square tilings.** Let $\mathbb{T}(\mathcal{P})$ be the foliated space of all planar tilings $\mathcal{T}$ constructed from a finite set of prototiles $\mathcal{P}$. Let us recall that tiles are translational copies of a finite number of polygons, which are touching face-to-face. For any repetitive tiling $\mathcal{T} \in \mathbb{T}(\mathcal{P})$, the continuous hull of $\mathcal{T}$ is the minimal closed subset $\mathbb{X} = \overline{L}_{\mathcal{T}}$. The induced lamination is transversely modelled by the set X of elements of $\mathbb{X}$ with the origin as base point. If $\mathcal{T}$ is also aperiodic, then X is homeomorphic to the Cantor set.

Theorem 4.1 ([15]). *The continuous hull of any planar tiling is orbit equivalent to the continuous hull of a tiling whose tiles are marked squares.* □

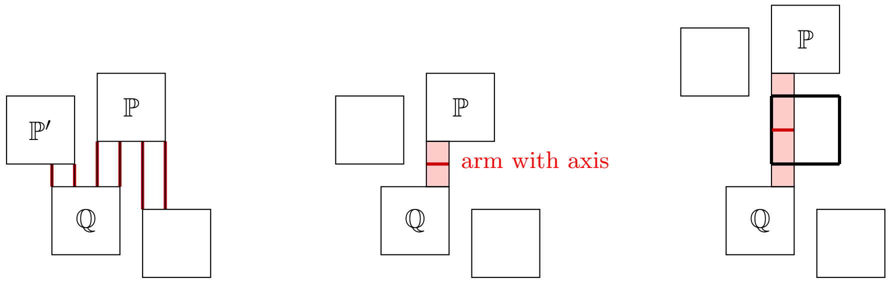
According to this theorem, we assume that $\mathcal{P}$ is a finite set of marked squares and we consider an aperiodic and repetitive tiling $\mathcal{T}$ in $\mathbb{T}(\mathcal{P})$. Furthermore, all leaves in the continuous hull $\mathbb{X}$ are endowed with the max-distance.

4.2. **Constructing an intermediate tiling.** Now, let us start by inflating some tiles in the usual sense: there is a finite number of of flow boxes $\mathbb{B}_{1,i} \cong \mathbb{P}_{1,i} \times C_{1,i}$, $i = 1, \ldots, k_1$, such that the plaques $\mathbb{P}_{1,i}$ are squares of side N_1 and the transversal $C_1 = \bigcup_{i=1}^{k_1} C_{1,i}$ is N_1-dense in $\mathbb{X}$ (i.e. any ball of radius N_1 meets C_1) with respect to the longitudinal max-distance. We say that $\mathcal{B}_1 = \{\mathbb{B}_{1,i}\}_{i=1}^{k_1}$ is a *partial box decomposition* of $\mathbb{X}$, and we write $\mathcal{P}_1 = \{\mathbb{P}_{1,i}\}_{i=1}^{k_1}$. The first step in order to inflate tilings in $\mathbb{X}$ is to replace $\mathcal{B}_1$ with a true box decomposition $\mathcal{B}'_1$. To do so, we need some preliminaries:

Definition 4.2. Let $\mathbb{P}$ and $\mathbb{Q}$ be two plaques of $\mathcal{B}_1$ contained in the same leaf of $\mathbb{X}$. We say that $\mathbb{Q}$ is a *neighbor of* $\mathbb{P}$ if the orthogonal distance from an edge of $\mathbb{P}$ to $\mathbb{Q}$ is the minimum of the distance from this edge of $\mathbb{P}$ to another plaque, see Figure 2(a). In this case, the union of the orthogonal segments which realize the distance between $\mathbb{P}$ and $\mathbb{Q}$ is called an *arm* (of the complement of the partial tiling defined by $\mathcal{B}_1$), see Figure 2(b). Such an arm contains a segment, called the *axis*, which is parallel and equidistant to the corresponding edges of $\mathbb{P}$ and $\mathbb{Q}$. Also note that $\mathbb{P}$ is not always a neighbor of $\mathbb{Q}$, even though $\mathbb{Q}$ is a neighbor of $\mathbb{P}$, see Figure 2(a).

Lemma 4.3. *Any arm has a length and width less than or equal to N_1.*

Proof. Obviously the length of any arm is bounded by the side length N_1 of the plaques of $\mathcal{B}_1$. On the other hand, if the width of an arm (between two plaques $\mathbb{P}$



(a) (b) (c)

FIGURE 2. Arms and axes

and $\mathbb{Q}$) were larger than N_1, there would be space for another plaque (between $\mathbb{P}$ and $\mathbb{Q}$) and C_1 would not be N_1-dense, see Figure 2(c). □

Lemma 4.4. *For each tiling $\mathcal{T} \in \mathbb{X}$, let us consider the union of the plaques of the partial box decomposition $\mathcal{B}_1$ and the corresponding arms. Each connected component of the complement of this union is a marked rectangle, called a* cross*, with sides of a length less than or equal to $2N_1$.*

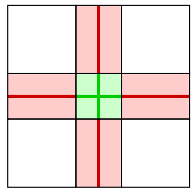
Proof. Since each plaque of $\mathcal{B}_1$ has at most 4 neighbors, the boundary of each connected component meets 3 or 4 plaques of $\mathcal{B}_1$ and 3 or 4 arms between these plaques. In both cases, the connected component is a rectangle of side length at most $2N_1$, as can be see in Figures 3 and 4. □

Remark 4.5. It is also interesting to note that there are degenerate arms and crosses, such as those illustrated in the Figure 5.

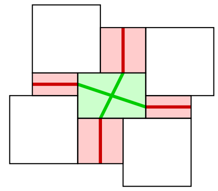
4.3. **Robinson inflation.** Using arms and crosses, we can replace the partial box decomposition $\mathcal{B}_1$ with a complete box decomposition $\mathcal{B}'_1$. In this step, we modify this decomposition so that it becomes inflated from the initial decomposition $\mathcal{B}$ by unit square plaques. We start by introducing some definitions:

Definition 4.6. i) The middle point of the intersection of each arm with a cross is called an *exit point* of the cross. Each side of a cross contains at most 1 exit point and each non-degenerate cross has at least 3 exit points, see Figures 3 and 4. They are *positive* or *negative* end points of the axes (relatively to the usual positive orientation). If the cross is degenerated, there are still at least 3 exit points counting multiplicities, see Figure 5.

ii) Each cross is decorated with a graph which is obtained by joining the center of mass with the exit points. In the non-degenerate case, this decoration separates the cross into 3 or 4 regions. In the degenerate case, the cross reduces to a common side of two different arms (decorated with the middle point contained in the two axes) or a single point (which is the intersection of the degenerate arms). Each of these regions is called a *cross-sector*. We also use the term *cross-sector* to refer to the union of $\mathcal{P}$-tiles which meet the original cross-sector. As explained in [3], it is irrelevant where $\mathcal{P}$-tiles which meet several sectors are included, although for simplicity we will always assume that such a $\mathcal{P}$-tiles are included in the sectors pointing right and upward.

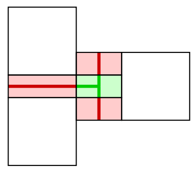


(a) Regular cross

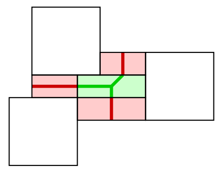


(b) Non-regular cross

Figure 3. Crosses with 4 arms



(a) Regular cross



(b) Non-regular cross

Figure 4. Crosses with 3 arms

iii) Let $\mathbb{P}$ be a plaque of the partial box decomposition $\mathcal{B}_1$. We denote by $\mathbb{P}'$ the union of $\mathbb{P}$ with the half-arms and the cross-sectors meeting $\mathbb{P}$. Replacing these half-arms and cross-sectors with corresponding half-arms and cross-sectors made of $\mathcal{P}$-tiles as above, we obtain another planar set $\mathbb{P}''$. Now $\mathbb{P}''$ is a tile of a planar tiling $\mathcal{T}''$ inflated from $\mathcal{T}$ and a plaque of a box decomposition $\mathcal{B}_1''$ inflated from $\mathcal{B}$. We call this process *Robinson inflation*.

Lemma 4.7. *Any tile $\mathbb{P}'$ and any inflated tile $\mathbb{P}''$ contain a square of side N_1, and they are contained in a square of side $3N_1$. Thus their areas* $\mathrm{area}(\mathbb{P}')$ *and* $\mathrm{area}(\mathbb{P}'')$ *are comprised between N_1^2 and $9N_1^2$. If $\mathbb{P}'$ and $\mathbb{P}''$ are associated to the same plaque $\mathbb{P}$ of the partial box decomposition $\mathcal{B}_1$, then*

$$\mathrm{area}(\mathbb{P}'') \geq \mathrm{area}(\mathbb{P}') - \mathrm{length}(\partial\mathbb{P}')\,\mathrm{area}(\mathbb{P}_0) = \mathrm{area}(\mathbb{P}') - \mathrm{length}(\partial\mathbb{P}')$$

and

$$\mathrm{length}(\partial\mathbb{P}'') \leq \mathrm{length}(\partial\mathbb{P}')\,\mathrm{length}(\partial\mathbb{P}_0) \leq 64N_1$$

where $\mathbb{P}_0$ is the square prototile of $\mathcal{T}$.

Proof. Firstly, since $\mathbb{P}''$ is obtained from $\mathbb{P}'$ by adding or removing $\mathcal{P}$-tiles, we have that

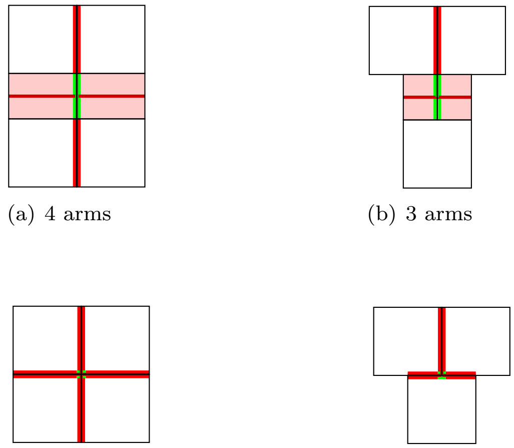
$$\mathrm{area}(\mathbb{P}'') \geq \mathrm{area}(\mathbb{P}') - \mathrm{length}(\partial\mathbb{P}')\,\mathrm{area}(\partial\mathbb{P}_0)$$

and

$$\mathrm{length}(\partial\mathbb{P}'') \leq \mathrm{length}(\partial\mathbb{P}')\,\mathrm{length}(\partial\mathbb{P}_0)$$

where $\mathrm{area}(\mathbb{P}_0) = 1$ and $\mathrm{length}(\partial\mathbb{P}_0) = 4$. On the other hand, when we replace $\mathbb{P}$ with $\mathbb{P}'$, the side length increases by at most $2\sqrt{2}N_1 \leq 3N_1$, where $2\sqrt{2}N_1$ is the product of the maximum number of non-degenerate crosses which meet each edge of $\mathbb{P}$ (equal to 2) and the half-diagonal of the square of side $2N_1$ (equal to $\sqrt{2}N_1$). Thus $\mathrm{length}(\partial\mathbb{P}') \leq 4(N_1 + 3N_1) = 16N_1$ and we have finished. □

Arguing inductively, we have the following theorem:



(a) 4 arms (b) 3 arms

(c) 4 arms (d) 3 arms

FIGURE 5. Degenerate crosses and arms

Theorem 4.8. *There is a sequence of box decompositions $\mathcal{B}''_n$ such that $\mathcal{B}''_0 = \mathcal{B}$ and $\mathcal{B}''_{n+1}$ is obtained by Robinson inflation of $\mathcal{B}''_n$.* □

We denote by $\mathcal{P}'_n$ and $\mathcal{P}''_n$ the sets of the plaques of $\mathcal{B}'_n$ and $\mathcal{B}''_n$ respectively. Then $\mathcal{B}'_n$ and $\mathcal{B}''_n$ induce tilings $\mathcal{T}'_n \in \mathbb{T}(\mathcal{P}'_n)$ and $\mathcal{T}''_n \in \mathbb{T}(\mathcal{P}''_n)$ on each leaf of $\mathbb{X}$. By construction, each plaque $\mathbb{P}'_{n+1}$ of the box decomposition $\mathcal{B}'_{n+1}$ is a $\mathcal{P}'_n$-patch, which is constructed from a square of side N_{n+1} and contained in a square of side $3N_{n+1}$. In the same way, each plaque $\mathbb{P}''_{n+1}$ of the inflated box decomposition $\mathcal{B}''_{n+1}$ is a $\mathcal{P}''_n$-patch. Thus, the tiling $\mathcal{T}''_{n+1} \in \mathbb{T}(\mathcal{P}''_{n+1})$ induced by $\mathcal{B}''_{n+1}$ is obtained by Robinson inflation from the tiling $\mathcal{T}''_n \in \mathbb{T}(\mathcal{P}''_n)$ induced by $\mathcal{B}''_n$. As in the proof of Lemma 4.7, we can see that

$$\operatorname{area}(\mathbb{P}''_{n+1}) \geq \operatorname{area}(\mathbb{P}'_{n+1}) - \operatorname{length}(\partial \mathbb{P}'_{n+1}) A_n$$

and

$$\operatorname{length}(\partial \mathbb{P}''_{n+1}) \leq \operatorname{length}(\partial \mathbb{P}'_{n+1}) L_n$$

where $A_n = \max\{\operatorname{area}(\mathbb{P}''_n) \mid \mathbb{P}''_n \in \mathcal{P}''_n\}$ and $L_n = \max\{\operatorname{length}(\mathbb{P}''_n) \mid \mathbb{P}''_n \in \mathcal{P}''_n\}$. On the other hand, we have that

$$\operatorname{area}(\mathbb{P}'_{n+1}) \geq N_{n+1}^2 \quad \text{and} \quad \operatorname{length}(\partial \mathbb{P}'_{n+1}) \leq 16 N_{n+1}.$$

Finally, we can assume that $L_n \leq A_n$ when n is chosen large enough.

Proposition 4.9. *There is a sequence of positive integers N_n such that the isoperimetric ratio*

$$\frac{\operatorname{length}(\partial \mathbb{P}''_n)}{\operatorname{area}(\mathbb{P}''_n)} \to 0$$

as $n \to \infty$.

Proof. According to the previous inequalities, if we choose $N_{n+1} \geq N_n^3$, then the isoperimetric ratio

$$\begin{aligned}\frac{\operatorname{length}(\partial \mathbb{P}''_{n+1})}{\operatorname{area}(\mathbb{P}''_{n+1})} &\leq \frac{\operatorname{length}(\partial \mathbb{P}'_{n+1}) L_n}{\operatorname{area}(\mathbb{P}'_{n+1}) - \operatorname{length}(\partial \mathbb{P}'_{n+1}) A_n} \\ &\leq \frac{\operatorname{length}(\partial \mathbb{P}'_{n+1}) A_n}{\operatorname{area}(\mathbb{P}'_{n+1}) - \operatorname{length}(\partial \mathbb{P}'_{n+1}) A_n} \\ &= \frac{1}{\frac{\operatorname{area}(\mathbb{P}'_{n+1})}{\operatorname{length}(\partial \mathbb{P}')} \frac{1}{A_n} - 1} \\ &\leq \frac{1}{\frac{N_{n+1}^2}{16 N_{n+1}} \frac{1}{9N_n^2} - 1} = \frac{(12N_n)^2}{N_{n+1} - (12N_n)^2}\end{aligned}$$

converges to 0. □

5. Properties of the boundary for Robinson inflation

In this section, we replace the initial box decompositions $\mathcal{B}^{(n)}$ by the box decompositions $\mathcal{B}''_n$ provided by Theorem 4.8, but we keep all of the notations introduced in Section 3. Thus, according to Proposition 3.2, we have a minimal open AF equivalence subrelation $\mathcal{R}_\infty = \varinjlim \mathcal{R}_n$ of the equivalence relation $\mathcal{R}$. Our first aim is to prove Proposition 3.6:

Proof of Proposition 3.6. Let μ be a $\mathcal{R}_\infty$-invariant probability measure on X. For each discrete flow box $B''_n \cong P''_n \times C''_n$ belonging to $\mathcal{B}''_n$, we have that

$$\mu(B''_n) = \#P''_n \mu(C''_n) \quad \text{and} \quad \mu(\partial_v B_n) = \#\partial P''_n \mu(C''_n)$$

where $\#P''_n$ and $\#\partial P''_n$ denote the number of elements of the discrete plaque P''_n and its boundary $\partial P''_n$. For each positive integer $n \geq 1$, it follows that

$$\begin{aligned}\mu(\partial \mathcal{R}_n) &= \sum_{\mathbb{B}''_n \in \mathcal{B}''_n} \mu(\partial_v B''_n) \\ &= \sum_{\mathbb{B}''_n \in \mathcal{B}''_n} \#\partial P''_n \mu(C''_n) \\ &= \sum_{\mathbb{B}''_n \in \mathcal{B}''_n} \frac{\#\partial P''_n}{\#P''_n} \mu(B''_n) \\ &\leq \max_{\mathbb{P}''_n \in \mathcal{P}''_n} \left\{ \frac{\#\partial P''_n}{\#P''_n} \right\} \sum_{\mathbb{B}''_n \in \mathcal{B}''_n} \mu(B''_n) \\ &= \max_{\mathbb{P}''_n \in \mathcal{P}''_n} \left\{ \frac{\#\partial P''_n}{\#P''_n} \right\} \mu(X) = \max_{\mathbb{P}''_n \in \mathcal{P}''_n} \left\{ \frac{\#\partial P''_n}{\#P''_n} \right\}\end{aligned}$$

Since the leaves of the continuous hull $\mathbb{X}$ are quasi-isometric to the $\mathcal{R}$-equivalence classes in X, Proposition 4.9 implies that

$$\lim_{n \to \infty} \frac{\#\partial P''_n}{\#P''_n} = 0$$

for all $\mathbb{P}''_n \in \mathcal{P}''_n$. In fact, by replacing the max-metric along the leaves with the discrete metric (defined as the minimum length of the paths of $\mathcal{P}$-tiles connecting two

points), we can assume that length($\partial\mathbb{P}''_n$) = $\#\partial P''_n$ and area($\mathbb{P}''_n$) = $\#P''_n$. Anyway, we have that

$$\mu(\partial\mathcal{R}_\infty) = \lim_{n\to\infty} \mu(\partial\mathcal{R}_n) = 0$$

and so $\partial\mathcal{R}_\infty$ is $\mathcal{R}_\infty$-thin. □

As announced, we are interested in the following key property of the boundary:

Proposition 5.1. *Any $\mathcal{R}$-equivalence class separates into at most four $\mathcal{R}_\infty$-equivalence classes.*

We intend to demonstrate that each acyclic graph $\Gamma = \Gamma_{\mathcal{T}}$ contained in the continuous boundary $\partial_c\mathcal{R}_\infty$ is actually a tree which separates the corresponding leaf $L = L_{\mathcal{T}} \subset \mathbb{X}$ into at most 4 connected components. Firstly, in order to prove Proposition 5.1, we need to introduce some definitions and distinguish some cases. Let us recall that we have a sequence of box decompositions $\mathcal{B}'_n$ of $\mathbb{X}$ whose plaques are obtained from squares $\mathbb{P}_n$ of side N_n, arms A_n and crosses C_n. In the next, we will always assume that all these plaques belong to the same leaf L of $\mathbb{X}$.

Definition 5.2. We call *virtual arm* of the inflation process a sequence of arms A_n whose axes a_n are contained in a horizontal or vertical ribbon of constant width. Likewise, a *virtual cross* is a sequence of crosses C_n whose centers of mass c_n are contained in a square of constant side. Note that the axes and centers of mass can oscillate in the interior of the ribbon or square, see Figure 6.

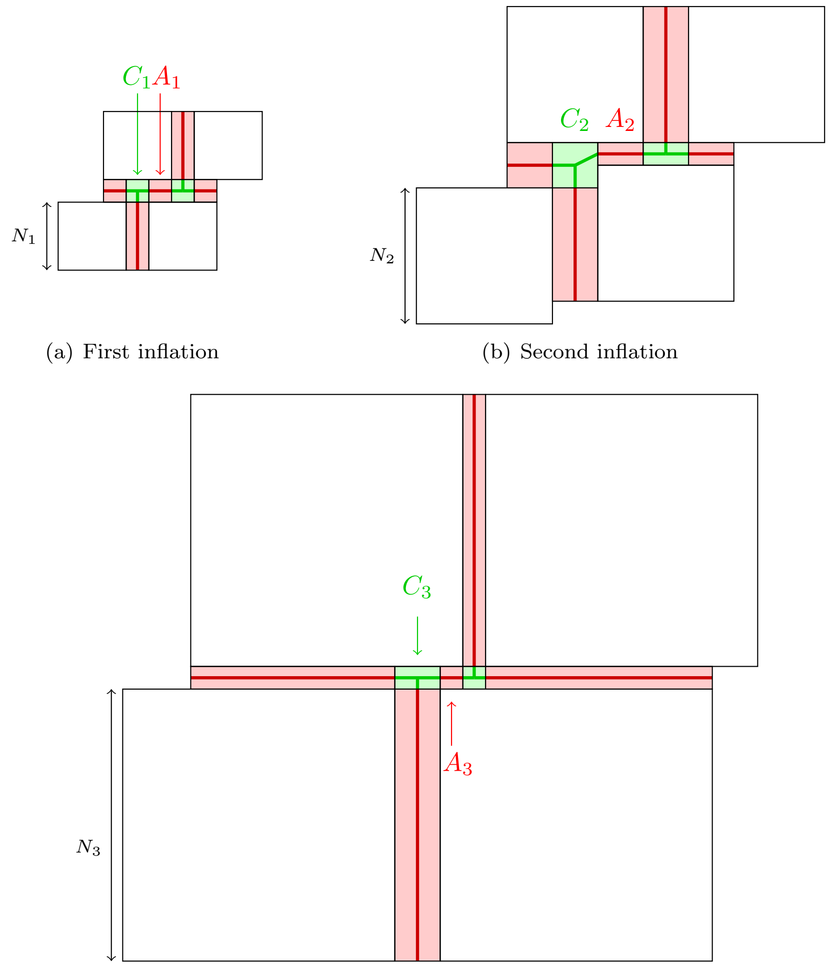
Proof of Proposition 5.1. Firstly, we give a more accurate version of Proposition 3.5. Recall that the continuous boundary $\partial_c\mathcal{R}_\infty$ intersect L in an acyclic graph Γ without terminal edges. But Γ is actually a tree when we use Robinson inflation. Indeed, if we assume on the contrary that Γ is not connected, then there would be a sequence $\mathbb{P}''_n$ of inflated $\mathcal{P}''_n$-tiles such that the distance between two disjoint edges would be bounded. But it is not possible in our case. Secondly, we distinguish some cases:

Case 1) *There are neither axes nor virtual crosses.* In this case, L does not intersect $\partial_c\mathcal{R}_\infty$. This means that $\mathcal{R}_\infty[\mathcal{T}] = \mathcal{R}[\mathcal{T}]$ for all $\mathcal{T} \in L \cap X$.

Case 2) *There is a virtual axis, but there are no virtual crosses.* By definition, there is a sequence of arms A_n whose axes a_n remain in the interior of a ribbon of constant width. There are now two possibilites:

Subcase 2.1) *The positive and negative end points of the axes a_n do not remain in the interior of a square of constant side.* Thus, all the axes a_n grow in both opposite (horizontal or vertical) directions determined by the axis of the ribbon. For simplicity, we say that the axes a_n grow *on the left and right* in the horizontal case and *upward and downward* in the vertical case. In both cases, since $\partial_c\mathcal{R}_\infty$ intersect L in a tree without terminal edges (see Proposition 3.5), which is contained in the horizontal or vertical ribbon, this intersection separates L into 2 connected components, and therefore $\mathcal{R}[\mathcal{T}]$ separates into two $\mathcal{R}_\infty$-equivalence classes.

Subcase 2.2) *Positive or negative end points remain in the interior of a square of constant side.* Let us assume first that only negative end points remain in the interior of a square of constant side. In this case, since there are no virtual crosses, there are bigger and bigger crosses C_n whose exit points on the right (resp. up) side coincide with the negative end points of the axes a_n of the horizontal (resp. vertical) arms A_n. Thus, the decorations of the crosses C_n must grow on the left



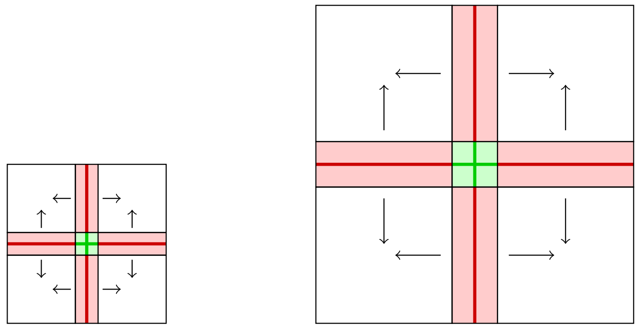
(a) First inflation

(b) Second inflation

(c) Third inflation

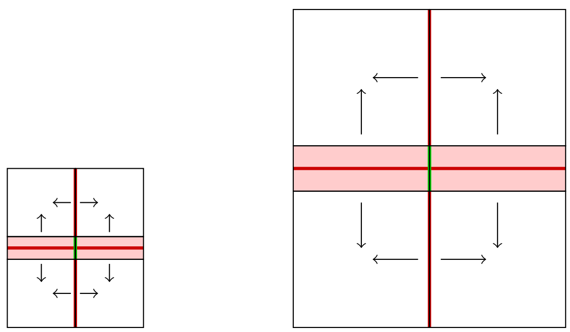
FIGURE 6. Virtual arms and crosses

(i.e. the negative direction of the axis of the horizontal ribbon) or downward (i.e. the negative direction of the axis of the vertical ribbon). Now, by joining the axis a_n with the horizontal (resp. vertical) edge of the decoration of C_n, this subcase is reduced to the above one. A similar argument may be used if only positive end points remain in the interior of a square of constant side. Finally, if both negative and positive end points remain in the interior of such a square, then there is a virtual arm (defined by a sequence of arms of bounded lenght) connecting bigger and bigger crosses C_n and C'_n on the left and right (resp. upward and downward) side. They must have 3 exit points, and grow in the opposite directions. Arguing on the left and right (resp. upward and downward) as in the previous cases, we come to the same conclusion.



(a) First inflation (b) Second inflation

FIGURE 7. Virtual cross with 4 exit points

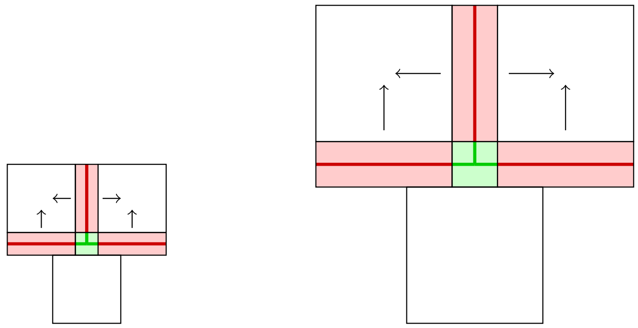


(a) First inflation (b) Second inflation

FIGURE 8. Virtual cross with degenerate arms

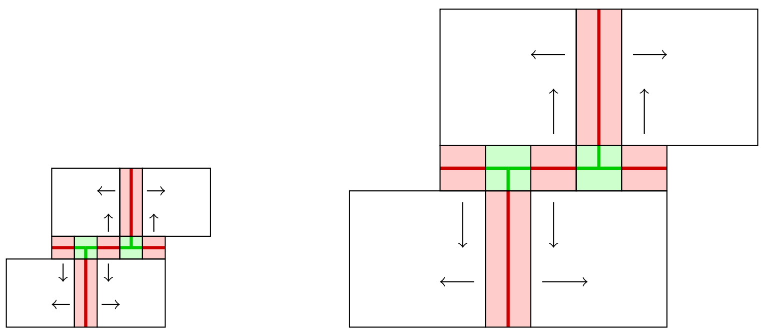
Case 3) *There is a single virtual cross.* Thus, there is a sequence of crosses C_n whose centers of mass c_n belong to a square of constant side. As before, we distinguish two subcases:

Subcase 3.1) *The crosses have four exit points (including multiplicities).* Assume first that the crosses grow in the horizontal (on the left and right) and vertical (upward and downward) opposite directions, see Figure 7. Since the crosses cover the whole leaf L, the continuous boundary $\partial_c \mathcal{R}_\infty$ separates L into 4 connected components, likewise decorations separates crosses. In general, the exit points located on the left and right sides of the crosses are positive and negative end points of the axes of horizontal arms pointing to the left and right. Similarly the exit points on the up and down sides are negative and positive end points of the axes of vertical arms pointing upward and downward. Thus, if the crosses do not grow in all directions, there will be exit points on the right or left, up or down, remaining in the interior of a square of constant side. But these points will be end points of the axes defining virtual arms pointing to the left or right, upward or downward. From the discussion of the previous case, it is clear that $\partial_c \mathcal{R}_\infty$ still separates L into 4 connected components. On the other hand, we may have a degenerate virtual cross defined by a sequence of degenerate crosses of type (a) or (c) in Figure 5. In



(a) First inflation (b) Second inflation

FIGURE 9. Virtual cross with 3 exit points



(a) First inflation (b) Second inflation

FIGURE 10. Two virtual crosses

this case, there will always be 4 virtual arms, while some will be degenerate, see Figure 8. In any event, as before, $\mathcal{R}[\mathcal{T}]$ separates into four $\mathcal{R}_\infty$-equivalence classes.

Subcase 3.2) *The crosses have three exit points (including multiplicities).* This situation is obviously very similar to the previous one, see Figure 9. Arguing as before, we can see that $\partial_c \mathcal{R}_\infty$ separates L into 3 connected components, and then $\mathcal{R}[\mathcal{T}]$ separates into three $\mathcal{R}_\infty$-equivalence classes.

Before dealing with the last case, we would like to point out that there may actually be different types of configurations in sucessive inflation steps. In fact, since $\mathcal{B}'_{n+1}$ is not necessarily inflated from $\mathcal{B}'_n$, we may have a virtual cross defined by a sequence of crosses whose number of exit points (including multiplicities) is not stationary. But because $\mathcal{B}''_{n+1}$ is inflated from $\mathcal{B}''_n$, the number of $\mathcal{R}_\infty$-classes in each $\mathcal{R}$-class $\mathcal{R}[\mathcal{T}]$ is determined by the configuration with the fewest number of exit points.

Case 4) *There is more than one virtual cross.* Let us recall that the side of the plaques of the partial decomposition $\mathcal{B}_n$ tend to ∞. So the only possibility is that there are two virtual crosses defined by sequences of crosses with 3 exit points (which may be non-degenerate or degenerate of type (b) or (d) as in Figure 5) connected

by horizontal (or vertical) arms of bounded length and whose other horizontal (resp. vertical) arms grow in opposite directions. Furthemore, their only vertical (resp. horizontal) arms must also grow pointing in opposite directions, see Figure 10. In this case, $\partial_c\mathcal{R}_\infty$ still separates L into 4 connected components, and then $\mathcal{R}[\mathcal{T}]$ separates into four $\mathcal{R}_\infty$-equivalence classes. □

Let us resume the previous results in the following statement:

Proposition 5.3. *Let $\mathbb{X}$ be the continuous hull of an aperiodic and repetitive planar tiling whose tiles are marked squares, and X the total transversal defined by the choice of base points in the prototiles. Let $\mathcal{R}$ be the EER on X induced by the natural lamination of $\mathbb{X}$, and $\mathcal{R}_\infty$ the minimal open AF equivalence subrelation defined by Robinson inflation. Then the boundary $\partial\mathcal{R}_\infty$ is a $\mathcal{R}_\infty$-thin meager closed subset of X such that every $\mathcal{R}$-equivalence class represented by a point of $\partial\mathcal{R}_\infty$ separates into at most four $\mathcal{R}_\infty$-equivalence classes.* □

6. Filtration and absorption

In this section, we complete the new proof of the Affability Theorem by replacing the following filtration:

$$\partial\mathcal{R}_\infty = \bigcup_{i=2}^{4} \partial_i\mathcal{R}_\infty = \bigcup_{i=2}^{4} \{\, \mathcal{T} \in X \mid \mathcal{R}[\mathcal{T}] \text{ is union of } i \ \mathcal{R}_\infty\text{-equivalence classes} \,\}$$

with a suitable filtration by closed subsets and applying the Absorption Theorem of [6] (see Theorem 3.7).

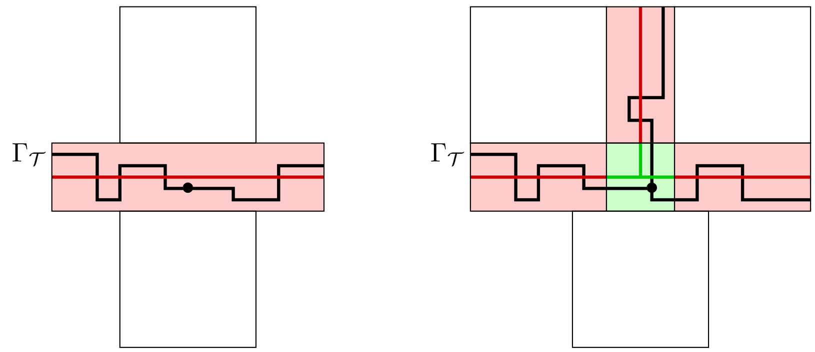
Lemma 6.1. *The subset $\mathcal{H}_2 = \partial_2\mathcal{R}_\infty$ of X is closed.*

Proof. Let us assume that $\mathcal{T}_n$ is a sequence of tilings belonging to $\partial_2\mathcal{R}_\infty$ which converges to a tiling $\mathcal{T}$. Since $\mathcal{R}[\mathcal{T}_n]$ splits into two different $\mathcal{R}_\infty$-equivalence classes, there are two different $\mathcal{P}''_n$-tiles which meet along a portion of $\Gamma_{\mathcal{T}_n}$. Recall that $\Gamma_{\mathcal{T}_n}$ is the tree of the partition of the continuous boundary $\partial_c\mathcal{R}_\infty$ passing through $\mathcal{T}_n$. Using the definition of the Gromov-Hausdorff topology for $R = N_n$, with a large enough n, we can deduce that the limit $\mathcal{T}$ contains enormous patches (which are the union of the two $\mathcal{P}''_n$-tiles above) with the same property, see Figure 11(a). This means that $\mathcal{T} \in \partial_2\mathcal{R}_\infty$. □

Contrary to the case of $\partial_2\mathcal{R}_\infty$, the subsets $\partial_3\mathcal{R}_\infty$ and $\partial_4\mathcal{R}_\infty$ are not longer closed. But we will construct countable families of closed fundamental domains for their $\mathcal{R}$-saturations. Let us start by recalling this definition:

Definition 6.2. A subset A of X is said to be a *fundamental domain* for the EER induced on its $\mathcal{R}$-saturation when A intersects all the $\mathcal{R}$-equivalence classes of this saturation in exactly one point.

Now, we start by replacing the subsets $\partial_3\mathcal{R}_\infty$ and $\partial_4\mathcal{R}_\infty$ of $\partial\mathcal{R}_\infty \subset X$ with the corresponding subsets $\check{\partial}_3\mathcal{R}_\infty$ and $\check{\partial}_4\mathcal{R}_\infty$ of $\check{\partial}\mathcal{R}_\infty = \partial_c\mathcal{R}_\infty \cap \check{X}$ where $\check{X}$ is the total transversal described in §3.1. For each tiling $\mathcal{T} \in \check{\partial}_3\mathcal{R}_\infty \cup \check{\partial}_4\mathcal{R}_\infty$, the sequence of tilings $\mathcal{T}'_n$ (induced by the box decomposition $\mathcal{B}'_n$) contains a sequence of crosses C_n with 3 or 4 arms defining a virtual cross. On the other hand, let us recall that $\partial_c\mathcal{R}_\infty$ is equipped with a natural equivalence relation induced by $\mathcal{F}$ whose classes are infinite acyclic graphs without terminal edges. According to the proof of



(a) Two ends (b) Three ends

FIGURE 11. Two and three ends

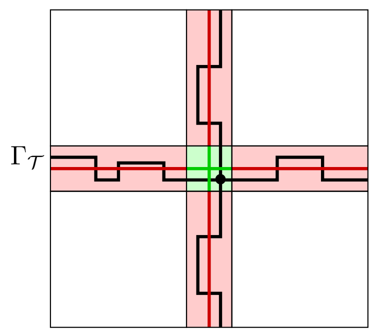


FIGURE 12. Four ends

Proposition 5.1, they are actually trees with 2, 3 or 4 ends, see Figures 11(b), 12, and 13. Let $\Gamma = \Gamma_{\mathcal{T}}$ be the tree passing through $\mathcal{T}$. By definition, Γ is the intersection of infinite graphs $\Gamma^{(n)}$ obtained from the edges of the inflated tilings $\mathcal{P}''_n$. For each $n \geq 1$, there are 3 or 4 different $\mathcal{P}''_n$-tiles which meet in a neighborhood in $\Gamma^{(n)}$ of some fixed vertex $o = o_\Gamma$ in C_n. They form a $\mathcal{P}''_n$-patch $\mathbb{M}_n(\Gamma)$.

Definition 6.3. We say that o is the *root* of Γ and $\mathbb{M}_n(\Gamma)$ is a *basic $\mathcal{P}''_n$-patch around the root o*. We denote by $D(\mathbb{M}_n(\Gamma))$ the number of $\mathcal{P}''_n$-tiles of $\mathbb{M}_n(\Gamma)$.

Remark 6.4. Notice that the union of the decoration of C_n and the axis of the corresponding arms A_n forms a rough model for the finite tree $\Gamma_n = \Gamma \cap \mathbb{M}_n(\Gamma)$. Axis of virtual arms provide (rough) global models for the ends of Γ, but in general these models reduce to a neighborhood of the root.

Lemma 6.5. *The natural EER induced on the $\mathcal{R}$-saturation of $\check{\partial}_3\mathcal{R}_\infty$ admits a fundamental domain $\check{\mathcal{H}}_3$ which is the union of countably many disjoint closed subsets $\check{\mathcal{H}}_{3,m}$.*

Proof. Given a tiling $\mathcal{T} \in \check{\partial}_3\mathcal{R}_\infty$, let $\Gamma = \Gamma_{\mathcal{T}}$ be the intersection of the continuous boundary $\partial\mathcal{R}_\infty$ and the leaf $L = L_{\mathcal{T}}$. According to the previous discussion (see again the proof of Proposition 5.1), we know that $\mathbb{M}_n(\Gamma)$ grows in at least three different directions, see Figure 11(b). But notice that we may also find a basic patch

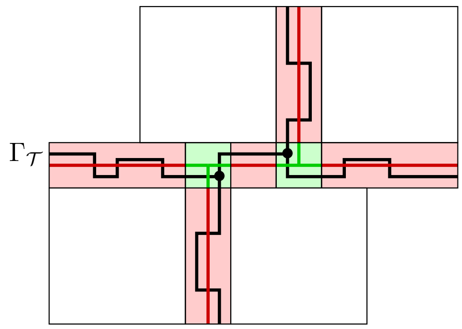


FIGURE 13. Four ends and two roots

of this kind in a leaf of the saturation of $\mathcal{T} \in \check{\partial}_4\mathcal{R}_\infty$ having exactly two virtual crosses. In this case, Γ has two roots o and o', see Figure 13. If we assume that $\mathbb{M}_n(\Gamma)$ contains a ball of radius n centered at o and if n is larger than the distance ℓ between o and o', we need 4 different $\mathcal{P}''_n$-tiles to cover the ball. In other words, for a large enough n, the basic $\mathcal{P}''_n$-patches around o and o' have non-trivial intersection, and we can replace each of the original basic $\mathcal{P}''_n$-patches (made up of 3 different $\mathcal{P}''_n$-tiles) with its union (made up of 4 different $\mathcal{P}''_n$-tiles). We still denote by $\mathbb{M}_n(\Gamma)$ the new basic $\mathcal{P}''_n$-patch around o and o'. Now, since o belongs to $\check{\partial}_3\mathcal{R}_\infty$, there is a minimal integer $m = m(\Gamma) \geq 0$ such that $D(\mathbb{M}_n(\Gamma)) = 3$ for all $n \geq m$.

Let $X_n(\Gamma)$ be the set of tilings in $\check{X}$ containing the patch $\mathbb{M}_n(\Gamma)$ (which cover the ball of radius n) around the origin. This is a clopen subset of $\check{X}$. Let us note that each clopen subset $X_n(\Gamma) \cap \check{\partial}\mathcal{R}_\infty$ is a fundamental domain for the equivalence relation induced on $\partial_c\mathcal{R}_\infty$. Indeed, even if $X_n(\Gamma) \cap \check{\partial}\mathcal{R}_\infty$ meets a leaf (which belong to the saturation of $\mathcal{T} \in \check{\partial}_4\mathcal{R}_\infty$) having exactly two virtual crosses, it meets just once, see again Figure 13. But $X_n(\Gamma)$ is never a fundamental domain for the EER induced on the $\mathcal{R}$-saturation of $\check{\partial}_3\mathcal{R}_\infty$. However, the intersection

$$\check{\mathcal{H}}_{3,\Gamma} = \bigcap_{n \geq m} X_n(\Gamma)$$

is a closed subset of $\check{\partial}_3\mathcal{R}_\infty$ which meets all the $\mathcal{R}$-equivalence classes in the $\mathcal{R}$-saturation of $\check{\partial}_3\mathcal{R}_\infty$ at most one time, although its $\mathcal{R}$-saturation may be smaller than that of $\check{\partial}_3\mathcal{R}_\infty$. But there are uncountable many finite labeled trees Γ with 3 ends. However, we can consider another closed set

$$\check{\mathcal{H}}_{3,\Gamma_m} = \bigcap_{n \geq m} \bigcup_{\Gamma'_m = \Gamma_m} X_n(\Gamma')$$

where Γ' represents any tree in $\partial_c\mathcal{R}_\infty$ such that $\mathbb{M}_m(\Gamma') = \mathbb{M}_m(\Gamma)$ and therefore $\Gamma'_m = \Gamma' \cap \mathbb{M}_m(\Gamma') = \Gamma \cap \mathbb{M}_m(\Gamma) = \Gamma_m$. Since for each $m \in \mathbb{N}$, there are only finite many trees Γ_m, the union $\check{\mathcal{H}}_{3,m}$ of the closed sets $\check{\mathcal{H}}_{3,\Gamma_m}$ is again a closed set. Their union

$$\check{\mathcal{H}}_3 = \bigcup_{m \in \mathbb{N}} \check{\mathcal{H}}_{3,m}$$

is fundamental domain which meets each leaf of the saturation of $\check{\partial}_3\mathcal{R}_\infty$ in a unique point: the root o of its boundary Γ. $\square$

Remarks 6.6. i) In the proof above, we have actually constructed a continuous map D from the subset of roots in $\check{\partial}_{\geq 3}\mathcal{R}_\infty = \check{\partial}_3\mathcal{R}_\infty \cup \check{\partial}_4\mathcal{R}_\infty$ to the Cantor set $\{3,4\}^{\mathbb{N}}$. This subset can be described as the inverse limit $\varprojlim \mathfrak{M}_n$ where $\mathfrak{M}_n$ is the set of the discrete basic patches $M_n = \mathbb{M}_n \cap \check{\partial}\mathcal{R}_\infty$ around any vertex of degree 3 or 4 and the obvious onto map $\mathfrak{M}_n \to \mathfrak{M}_{n-1}$ is given by the natural division of the inflated patches. Then $\check{\mathcal{H}}_3$ is the inverse image by D of the set of all sequences s_n which are cofinals to the constant sequence 3.

ii) From the lemma, going back to the original total transversal X, we can obtain 3 disjoint fundamental domains for the $\mathcal{R}$-saturation of $\partial_3\mathcal{R}_\infty$ which are related by partial transformations of $\mathcal{R}$. We denote by $\mathcal{H}_3$ any of these fundamental domains.

The construction of a fundamental domain for the $\mathcal{R}$-saturation of $\check{\partial}_4\mathcal{R}_\infty$ should be a little different since many leaves admit two roots. Only in the case when the leaves have a unique real root of degree 4, see Figure 12, we can argue as in the previous case. The rest of four-divided leaves (with two roots of degree 3 as in Figure 13) should be treated in other way.

Lemma 6.7. *There is a closed subset $\check{\mathcal{H}}_{4,0} \subset \check{\partial}_4\mathcal{R}_\infty$ of the continuous boundary $\check{\partial}\mathcal{R}_\infty$ intersecting all $\mathcal{R}$-equivalence classes in at most one point: the unique vertex of degree 4 of the corresponding tree in $\partial_c\mathcal{R}_\infty$.* □

Remark 6.8. As above, using this lemma, we obtain 4 disjoint closed fundamental domains for the $\mathcal{R}$-saturation of $\partial_4\mathcal{R}_\infty$ which are related by partial transformations of $\mathcal{R}$. Any of these fundamental domains will be denoted by $\mathcal{H}_{4,0}$.

In general, for each tiling $\mathcal{T} \in \check{\partial}_4\mathcal{R}_\infty - \check{\mathcal{H}}_{4,0}$, we have pairs of crosses C_n and C'_n with 3 exit points defining two virtual crosses connected by a virtual arm of bounded length. Let us assume that these crosses are of the type described in Figure 13, that is, the correspondings arms are included in the union of a horizontal ribbon and two vertical semi-ribbons pointing upward and downward. Like for tilings in $\check{\partial}_3\mathcal{R}_\infty$, the union of the decoration of C_n and C'_n and the axis of the corresponding arms A_n and A'_n forms a rough model for the tree $\Gamma = \Gamma_{\mathcal{T}}$ passing through $\mathcal{T}$ in a neighborhood of the two roots o and o', see Figure 13. In this case, the axes of the vertical semi-ribbons meet the horizontal axis in two different points, making up a (rough) global model for Γ. We define $\check{\mathcal{H}}_{4,\ell}$ as the set of tilings $\mathcal{T} \in \check{\partial}_4\mathcal{R}_\infty$ with origin in a o such that the distance to the other root o' is equal to a nonnegative integer ℓ.

Lemma 6.9. *The natural EER induced on the $\mathcal{R}$-saturation of $\check{\partial}_4\mathcal{R}_\infty$ admits a fundamental domain $\check{\mathcal{H}}_4$ which is the union of countably many disjoint closed subsets $\check{\mathcal{H}}_{4,\ell}$.*

Proof. We start by fixing a positive integer $\ell \geq 1$, and considering a tiling $\mathcal{T} \in \check{\partial}_4\mathcal{R}_\infty$ such that there is a geodesic path γ of length l joining o and o'. If $n \geq 0$ is large enough, there are 4 different $\mathcal{P}''_n$-tiles which meet in a neighborhood of γ in Γ. As in the proof of Lemma 6.5, they form a basic $\mathcal{P}''_n$-patch around any vertex of γ which coincides with $\mathbb{M}_n(\Gamma)$. In the same way, we denote by $X_n(\Gamma)$ the set of tilings in $\check{X}$ having this patch around the origin (that becomes one of the vertices of γ). In this case, since all the points of γ belong to $\check{\partial}_4\mathcal{R}_\infty$, there is a minimal integer $m = m(\gamma) \geq 0$ such that $D(\mathbb{M}_n(\Gamma)) = 4$ for all $n \geq m$. Therefore, if Γ' represents

any tree in $\partial_c\mathcal{R}_\infty$ such that $\mathbb{M}_m(\Gamma') = \mathbb{M}_m(\Gamma)$, then

$$\check{\mathcal{G}}_{4,\gamma} = \bigcap_{n\geq m} \bigcup_{\Gamma'_m=\Gamma_m} X_n(\Gamma')$$

is a closed subset of $\check{\partial}_4\mathcal{R}_\infty$ which meets all the $\mathcal{R}$-equivalence classes of the $\mathcal{R}$-saturation of $\check{\partial}_4\mathcal{R}_\infty$ in at most $\ell+1$ points. But since there is a finite number of paths of length $\leq \ell$ starting from the origin, the union $\check{\mathcal{G}}_{4,\ell}$ of all these closed subsets $\check{\mathcal{G}}_{4,\gamma}$ of $\check{\partial}_4\mathcal{R}_\infty$ is also a closed subset of $\check{\partial}_4\mathcal{R}_\infty$ with the same property. On the other hand, each closed subset $\check{\mathcal{G}}_{4,\gamma}$ split into $\ell+1$ closed subsets intersecting all $\mathcal{R}$-equivalence classes in at most one point. We denote by $\check{\mathcal{H}}_{4,\gamma}$ any of these closed sets. As before, their union $\check{\mathcal{H}}_{4,\ell}$ still has the same property. Then we have that

$$\check{\mathcal{H}}_4 = \bigcup_{\ell\geq 0} \check{\mathcal{H}}_{4,\ell}$$

is a fundamental domain for the $\mathcal{R}$-saturation of $\check{\partial}_4\mathcal{R}_\infty$. □

Remark 6.10. Once again, the fundamental domain $\check{\mathcal{H}}_4$ for the $\mathcal{R}$-saturation of $\check{\partial}_4\mathcal{R}_\infty$ provides us with 4 disjoint fundamental domains in X for this $\mathcal{R}$-saturation. They are related by partial transformations of $\mathcal{R}$, and any of these fundamental domains (which will be denoted by $\mathcal{H}_4$) split into countably many closed subsets $\mathcal{H}_{4,\ell}$.

Proof of Theorem 1.1. Finally, we complete the proof by applying the Absorption Theorem (Theorem 3.7) to the $\mathcal{R}_\infty$-thin closed subsets $\mathcal{H}_2$, $\mathcal{H}_{3,m}$, and $\mathcal{H}_{4,\ell}$ of the boundary $\partial\mathcal{R}_\infty$ (see Proposition 5.3). First, let us remark that the equivalence relation $\mathcal{R}$ induces CEERs on $\mathcal{H}_2$, $\mathcal{H}_{3,m}$, and $\mathcal{H}_{4,\ell}$ which are tranverse to those induced by $\mathcal{R}_\infty$. For $\mathcal{H}_2$, we have already explained how to obtain a decomposition of $\mathcal{H}_2$ into two clopen subsets A and B and a transformation $\varphi : A \to B$ whose graph generates a CEER $\mathcal{K}_2$ which is transverse to $\mathcal{R}_\infty|_{\partial_2\mathcal{R}_\infty}$. In this case, the first version of the Absorption Theorem given in [9] allows us to see that $\mathcal{R}_\infty \vee \mathcal{K}_2$ is orbit equivalent to $\mathcal{R}_\infty$. For each $\mathcal{H}_{3,m}$, we have two partial transformations from $\mathcal{H}_{3,m}$ to the other two closed subsets of $\partial_3\mathcal{R}_\infty$ which meet all the $\mathcal{R}$-equivalence classes in at most one point (see Lemma 6.5). They generates a CEER $\mathcal{K}_{3,m}$ defined by a free $\mathbb{Z}_3$-action, which is transverse to $\mathcal{R}_\infty|_{\partial_3\mathcal{R}_\infty}$. In a similar way (see Lemmas 6.7 and 6.9), we obtain another family of CEERs $\mathcal{K}_{4,\ell}$ defined by a free $\mathbb{Z}_4$-action with the same property. Applying inductively the Absortion Theorem of [6], we can exhaust $\mathcal{R}$ by a countable family of affable increasing EERs

$$\mathcal{R}_\infty \vee \mathcal{K}_2 \vee \mathcal{K}_{3,1} \vee \cdots \vee \mathcal{K}_{3,m} \vee \mathcal{K}_{4,0} \vee \cdots \vee \mathcal{K}_{4,\ell}.$$

Therefore $\mathcal{R}$ is affable. □

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